Renormalization of fluctuating tilted hexatic membranes

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We consider the tilted hexatic Hamiltonian on the fluctuating membranes. A renormalization-group analysis leads us to find three critical regions; two correspond to the strong coupling regimes of the gradient cross coupling where we find the (anti)locked tilted hexatic to liquid phase transition, the other to the weak coupling regime where we find four phases; the unlocked tilted hexatic phase, the hexatic phase, the tilted phase, and the liquid phase. The crinkled-to-crumpled transition of the fluctuating tilted hexatic membranes is also described. [S1063-651X(97)50107-3]

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Lyotropic liquid-crystal systems show a variety of phases with different types of in-plane two-dimensional (2D) order. Among the most interesting are the tilted hexatic phases, which have quasi-long-range order in two order parameters (the orientation of the local bond and the direction of the local molecular tilt), but only short-range translational order. Recently, there has been considerable progress in understanding tilted hexatic phases on the rigid layered liquid crystals. Selinger and Nelson [1] have presented a Landau theory for transitions among tilted hexatic phases. They consider the tilt-bond interaction potential and find several different hexatic phases differing from each other in the relation between the local bond orientation and the local tilt direction depending on the interaction potential parameters. However, they consider only phase transitions between low temperature phases (tilted hexatic phases) on the rigid 2D plane, in which disclinations in the bond orientational angle field $\theta_6(\mathbf{u})$ and vortices in the tilt-angle field $\theta_1(\mathbf{u})$ can be neglected.

In this Rapid Communication, we present a Landau theory, without the tilt-bond interaction potential, for transitions from tilted hexatic phase to disordered liquid phase on the fluctuating membranes. We consider the fluctuating membranes with the tilt and the hexatic in-plane orders described by the order parameters $\psi_1 = e^{i\theta_1}$ and $\psi_6 = e^{i6\theta_6}$, respectively. The tilt and the hexatic orders are coupled to each other via a gradient cross coupling introduced by Nelson and Halperin [2]. Depending on this gradient cross coupling parameter, we find three critical regions in the phase space of the tilt stiffness K_1, the hexatic stiffness K_6, and the gradient cross coupling K_{16}; two correspond to the strong coupling and the other to the weak coupling. We also show that without the tilt-bond interaction potential there exist a couple of different tilted hexatic phases. Finally, we discuss the crumpling transition of the fluctuating membranes with the tilt and the hexatic in-plane orders.

We parametrize the membrane by its position vector as a function of standard Cartesian coordinates $\mathbf{x} = (x, y)$;

$$\mathbf{R}(\mathbf{x}) = (\mathbf{x}, h(\mathbf{x})), \tag{1}$$

where $h(\mathbf{x})$ measures the deviation from the flat surface. This is called a Monge gauge. Associated with $\mathbf{R}(\mathbf{x})$ is a metric tensor $g_{\alpha\beta}(\mathbf{x}) = \partial_\alpha \mathbf{R}(\mathbf{x}) \cdot \partial_\beta \mathbf{R}(\mathbf{x})$ and a curvature tensor $K_{\alpha\beta}(\mathbf{x})$ defined via $K_{\alpha\beta}(\mathbf{x}) = \mathbf{N}(\mathbf{x}) \cdot \partial_\alpha \partial_\beta \mathbf{R}(\mathbf{x})$, where

$\mathbf{N}(\mathbf{x})$ is the local unit normal to the surface. From the curvature tensor $K_{\alpha\beta}$, the mean curvature H, and the Gaussian curvature K, are defined as follows:

$$H = \tfrac{1}{2} g^{\alpha\beta} K_{\beta\alpha}, \quad K = \det g^{\alpha\lambda} K_{\lambda\beta}, \tag{2}$$

where $g^{\alpha\beta}$ is the inverse tensor of $g_{\alpha\beta}$ satisfying $g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta$. In the continuum elastic theory, the long-wavelength properties of a fluctuating membrane are described by the Helfrich-Canham Hamiltonian [3]

$$\mathcal{H}_{\text{HC}} = \frac{1}{2} \int d^2x \sqrt{g} (\kappa H^2 + \bar{\kappa} K + \sigma), \tag{3}$$

where $g = \det g_{\alpha\beta}$, κ is the bending rigidity, $\bar{\kappa}$ is the Gaussian rigidity, and σ is the tension of the membrane. The first term is the mean curvature energy, the second the Gaussian curvature energy, and the third the surface tension energy. We are mostly interested in free membranes for which the topology is fixed and the renormalized surface tension obtained by differentiating the total free energy $\mathcal{F}$ with respect to the total surface area $\mathcal{A}$ ($\sigma_R = \partial \mathcal{F}/\partial \mathcal{A}$), is zero. Therefore, we will ignore the Gaussian curvature energy due to the topological invariance (the Gauss-Bonnet theorem) [5] and the surface tension energy with the understanding that it is really present if we want to keep track of how σ_R actually becomes zero [6]. In the Monge gauge, the mean curvature energy for the geometric shape fluctuations becomes

$$\mathcal{H}_{\text{HC}} = \frac{1}{2} \kappa \int d^2x \sqrt{1 + (\nabla h)^2} \left[\nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + (\nabla h)^2}} \right) \right]^2. \tag{4}$$

We consider the Hamiltonian for the tilt and the hexatic orders on a fluctuating membrane [4],

$$\mathcal{H}_{\text{TH}} = \tfrac{1}{2} K_1 \int d^2x \sqrt{g} g^{\alpha\beta} (\partial_\alpha \theta_1 - A_\alpha)(\partial_\beta \theta_1 - A_\beta)$$

$$+ \tfrac{1}{2}(36 K_6) \int d^2x \sqrt{g} g^{\alpha\beta} (\partial_\alpha \theta_6 - A_\alpha)(\partial_\beta \theta_6 - A_\beta)$$

$$+ (6 K_{16}) \int d^2x \sqrt{g} g^{\alpha\beta} (\partial_\alpha \theta_1 - A_\alpha)(\partial_\beta \theta_6 - A_\beta), \tag{5}$$

in terms of the local bond-angle field $\theta_6(\mathbf{u})$ and the tilt-angle field $\theta_1(\mathbf{u})$. The constants multiplied to the stiffnesses (K_1, K_{16}, K_6) are introduced to show the symmetry in the recursion relations which will be shown later. The gauge field A_α describes how the basis vector rotates under parallel transport according to the Gaussian curvature of the surface [7]. For simplicity, we dropped the tilt-bond interaction potential. Effects of the tilt-bond interaction near the fixed points may change the phase diagram qualitatively and deserve further investigation.

Thus we have the full Hamiltonian $\mathcal{H} = \mathcal{H}_{HC} + \mathcal{H}_{TH}$ to describe a fluctuating tilted hexatic membrane. We follow Park and Lubensky's treatment of the topological defects on fluctuating surfaces [8], and obtain the tilted hexatic membrane partition function

$$\mathcal{Z} = \int \mathcal{D}\mathbf{R}\, \mathcal{D}\phi_1 \mathcal{D}\phi_6 e^{-\mathcal{L}}, \tag{6}$$

with the effective Hamiltonian which is the two-field sine-Gordon Hamiltonian coupled to each other via the off-diagonal propagator and coupled to the geometry fluctuations by linear coupling to the scalar curvature $\mathcal{R}$ which is twice the Gaussian curvature $\mathcal{R} = 2K$

$$\mathcal{L} = \frac{1}{2} \int d^2x \nabla \phi_\mu \mathbf{M}_{\mu\nu}^{-1} \nabla \phi_\nu + \frac{i}{2\pi} \int d^2x (\phi_1 + \phi_6)\mathcal{R}$$

$$- \frac{2y_1}{a^2} \int d^2x \cos\phi_1 - \frac{2y_6}{a^2} \int d^2x \cos\phi_6 + \mathcal{H}_{HC}, \tag{7}$$

where we set $\beta = (1/k_B T) = 1$ and

$$\mathbf{M}_{\mu\nu} = \begin{pmatrix} K_1 & K_{16} \\ K_{16} & K_6 \end{pmatrix}, \tag{8}$$

a is the short-distance cutoff, y_1 (y_6) is the fugacity of vortices (disclinations), and ϕ_1 (ϕ_6) is the conjugate field to vortices (disclinations). When $\sqrt{g}$ is expanded in terms of h, nonvanishing lowest order terms in h are irrelevant and $\sqrt{g}$ is dropped in Eq. (7). In order to establish the renormalization-group (RG) recursion relations for the tilted hexatic rigidities, (K_1, K_{16}, K_6), and the fugacities, (y_1, y_6), we study the renormalization of the two-point vertex functions $\Gamma^{(2)}_{\phi_\mu \phi_\nu}(q)$ for the effective Hamiltonian in Eq. (7). Using the sine-Gordon renormalization analysis by Park and Lubensky [9], we find three critical regions.

(1) The strong coupling regimes near the points $\mathcal{S}_\pm \equiv (K_1, K_{16}, K_6) = (2/\pi, \pm 2/\pi, 2/\pi)$; to leading order in the fugacities, we obtain

$$\frac{d}{dl}K_1 = -4\pi^3 (K_1 y_1 + K_{16} y_6)^2,$$

$$\frac{d}{dl}K_6 = -4\pi^3 (K_{16} y_1 + K_6 y_6)^2, \tag{9}$$

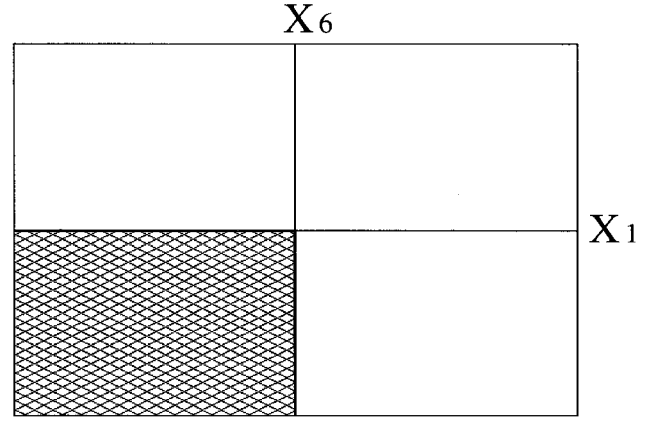$$\frac{d}{dl}K_{16} = -4\pi^3 (K_1 y_1 + K_{16} y_6)(K_{16} y_1 + K_6 y_6),$$

FIG. 1. Phase diagram in the strong coupling regime near $\mathcal{S}_\pm \equiv (K_1, K_{16}, K_6) = (2/\pi, \pm 2/\pi, 2/\pi)$. The shaded region is the antilocked (locked) tilted hexatic phase, outside the disordered liquid phase. At the antilocked (locked) tilted hexatic–liquid phase boundary, disclinations and vortices unbind simultaneously.

$$\frac{d}{dl}y_1 = (2 - \pi K_1)y_1, \quad \frac{d}{dl}y_6 = (2 - \pi K_6)y_6. \tag{10}$$

In the above equations, l is the renormalization-group parameter. As a check on these results, we set $K_1 = K_{16}^2 / K_6$ initially, and find that this self-duality condition is preserved under our renormalization transformation;

$$\frac{d}{dl}\left(K_1 - \frac{K_{16}^2}{K_6}\right) = 0 \tag{11}$$

as it should be. To study the system in the critical regions near the points $\mathcal{S}_\pm$, it is useful to introduce deviations defined by

$$K_1^{-1} = \pi/2(1 + X_1), \quad K_6^{-1} = \pi/2(1 + X_6) \tag{12}$$

$$K_{16}^{-1} = \pi/2(1 \pm X_{16}), \tag{13}$$

as well as rescaled fugacities

$$Y_1^2 = 8\pi^2 y_1^2, \quad Y_6^2 = 8\pi^2 y_6^2. \tag{14}$$

To lowest order in these variables, the recursion relations become

$$\frac{dX_1}{dl} = \frac{dX_{16}}{dl} = \frac{dX_6}{dl} = (Y_1 + Y_6)^2, \tag{15}$$

$$\frac{dY_1}{dl} = 2X_1 Y_1, \quad \frac{dY_6}{dl} = 2X_6 Y_6. \tag{16}$$

The flows generated by this system of the recursion relations are similar to the flows in the XY model [10]. The phase diagram in these regimes is shown in Fig. 1.

In the shaded region, the long-wavelength properties of the phase is described by the Hamiltonian in Eq. (5) with renormalized tilted hexatic stiffnesses (K_1^R, K_{16}^R, K_6^R). With the initial values $K_1(0) = K_6(0) = \pm K_{16}(0)$ and $Y_1(0) = Y_6(0)$, this effective Hamiltonian is minimized when two angle fields are antilocked (locked) by the constraint

$\nabla\theta_1 = \mp 6\nabla\theta_6$ in the mean field level, and taking into account thermal fluctuations the resulting state is the antilocked (locked) tilted hexatic phase near $\mathcal{S}_+$ ($\mathcal{S}_-$) in which the correlation functions show quasi-long-range order;

$$D_{16} \equiv \langle\psi_1(\mathbf{r})\psi_6(0)\rangle \simeq r^{-\eta_D} \quad \text{near} \quad \mathcal{S}_+ \qquad (17)$$

$$C_{16} \equiv \langle\psi_1(\mathbf{r})\psi_6^*(0)\rangle \simeq r^{-\eta_C} \quad \text{near} \quad \mathcal{S}_-, \qquad (18)$$

where the exponents η_C and η_D are related by

$$\eta_C = \eta_D = \frac{1}{2\pi K_1^R}. \qquad (19)$$

Disclinations and vortices are irrelevant in this region. All K_1, $|K_{16}|$, and K_6 are destabilized and pushed toward smaller values when either y_1 or y_6 starts to grow. This happens when $K_1 \lesssim 2/\pi$ or $K_6 \lesssim 2/\pi$, and we expect the antilocked (locked) tilted hexatic–liquid phase boundary when disclinations and vortices unbind simultaneously.

(2) The weak coupling regime near the point $\mathcal{W} \equiv (K_1, K_{16}, K_6) = (2/\pi, 0, 2/\pi)$; to leading order in the fugacities, we obtain

$$\frac{d}{dl}K_1 = -4\pi^3(K_1^2 y_1^2 + K_{16}^2 y_6^2),$$

$$\frac{d}{dl}K_6 = -4\pi^3(K_{16}^2 y_1^2 + K_6^2 y_6^2), \qquad (20)$$

$$\frac{d}{dl}K_{16} = -4\pi^3(K_1 K_{16} y_1^2 + K_6 K_{16} y_6^2),$$

$$\frac{d}{dl}y_1 = (2 - \pi K_1)y_1, \quad \frac{d}{dl}y_6 = (2 - \pi K_6)y_6. \qquad (21)$$

With the variables defined near the strong coupling fixed point except for K_{16} and defining $\bar{K}_{16} = (\pi/2)K_{16}$, we rewrite the system near the point $\mathcal{W}$ to lowest order,

$$\frac{dX_1}{dl} = Y_1^2 + \bar{K}_{16}^2 Y_6^2, \quad \frac{dX_6}{dl} = Y_6^2 + \bar{K}_{16}^2 Y_1^2, \qquad (22)$$

$$\frac{d\bar{K}_{16}}{dl} = -\bar{K}_{16}(Y_1^2 + Y_6^2), \qquad (23)$$

$$\frac{dY_1}{dl} = 2X_1 Y_1, \quad \frac{dY_6}{dl} = 2X_6 Y_6. \qquad (24)$$

Although the flows generated by this system are complicated, it is easy to check that the quantity

$$\mathcal{C} = 2X_1^2 + 2X_6^2 - 4X_1 X_6 \bar{K}_{16}^2 - (Y_1^2 + Y_6^2) \qquad (25)$$

is invariant to leading order along the trajectories,

$$\frac{d}{dl}\mathcal{C} = 0. \qquad (26)$$

Since $\mathcal{C}$ is entirely determined by Eq. (25) evaluated at $l = 0$, it is an analytic function of the initial conditions. Ac-
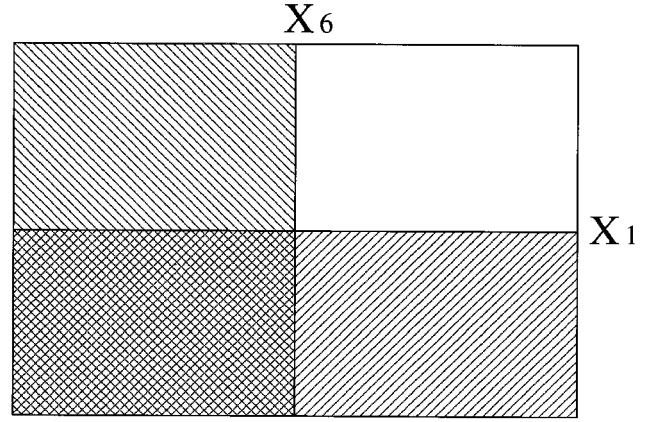

FIG. 2. Phase diagram in the weak coupling regime near $\mathcal{W} \equiv (K_1, K_{16}, K_6) = (2/\pi, 0, 2/\pi)$. Four phases—the unlocked tilted—hexatic phase (doubly shaded region), the tilted phase (shaded region with $X_1 < 0, X_6 > 0$), the hexatic phase (shaded region with $X_1 > 0, X_6 < 0$), and the liquid phase (unshaded region)—are shown. At the each phase boundary, disclinations and vortices unbind independently.

cording to the recursion formula (23), the space $K_{16} = 0$ is attractive and the long-wavelength properties are described by two independent sets of the XY-like renormalization recursion relations; one for the tilted-angle field, the other for the hexatic-angle field. The phase diagram in this regime with $K_{16} = 0$ is shown in Fig. 2.

K_1 is destabilized when y_1 starts to grow and this happens when $K_1 < 2/\pi$ and K_6 is destabilized when y_6 starts to grow and this happens when $K_6 < 2/\pi$, respectively. Since these instabilities occur independently, we expect there are four different phases; the unlocked tilted hexatic phase, the hexatic phase, the tilted phase, and the liquid phase. The phase boundaries are given by $K_1 = 2/\pi$ and $K_6 = 2/\pi$.

These two sets of the recursion relations are different from those derived by Nelson and Halperin. Their recursion relations seem to be in range of the small gradient cross coupling. Those are the same as the recursion relations in the weak coupling regime in this Rapid Communication except the recursion relation for the gradient cross coupling parameter. They claimed $dK_{16}/dl = 0$, and K_{16} remains fixed at its initial value to leading order in the fugacities. However, we find the recursion relation for the gradient cross coupling parameter nonvanishing in the weak coupling regime using the sine-Gordon field theoretic approach as well as the method of Kosterlitz [12] employed by Nelson and Halperin, and the space $K_{16} = 0$ in the phase space is attractive so that the initial nonvanishing value near the weak coupling critical point is pushed toward a smaller value. In addition, we find a completely different set of the recursion relations in the strong coupling regimes of the gradient cross coupling. In Kosterlitz's description, configurations of hybrid vortex-disclination pairs are created in addition to vortex antivortex and ± 1 disclination pairs to make the recursion relations in the strong coupling regimes different from those in the weak coupling regime where there is no logarithmic effective interaction between vortices and disclinations. Between the weak and strong coupling regimes along the K_{16} axis, there must be crossover regions near $K_{16} = \pm 1/\pi$ which are not accessible by the perturbation expansion employed in this

Rapid Communication. Higher order terms in the fugacities and the deviations from the critical points than the leading orders included in this Rapid Communication are necessary in the crossover regions [11]. To connect the renormalization flows in a topologically correct way in the weak and strong coupling regions, it is necessary to exploit the crossover region using the nonperturbative method.

To complete the RG recursion relations for the effective Hamiltonian $\mathcal{L}$, we study the renormalization of $\Gamma_{hh}^{(2)}(q)$ and obtain

$$\frac{d}{dl}\kappa = -\frac{3}{4\pi}\left[1 - \frac{1}{4\kappa}(K_1 + 2|K_{16}| + K_6)\right] \qquad (27)$$

in the strong coupling regime and

$$\frac{d}{dl}\kappa = -\frac{3}{4\pi}\left[1 - \frac{1}{4\kappa}(K_1 + K_6)\right] \qquad (28)$$

in the weak coupling regime. The renormalization of the bending rigidity is the same in both regimes except the appearance of K_{16} in the recursion relation for the strong coupling regime. Defining $K_1 + 2K_{16} + K_6 = k$ for the strong coupling regimes, we find a fixed line corresponding to the crinkled phase at $4\kappa = k$, and the crinkled-to-crumpled transition occurs at $k = 8/\pi$. The renormalized bending rigidity of the (anti-)locked tilted hexatic crinkled phase is $\kappa = 2/\pi$. In the weak coupling regime, there are three crinkled phases differing from each other in the internal ordering. The tilted and the hexatic crinkled phases has the renormalized bending rigidity $\kappa = 1/2\pi$ and the unlocked tilted hexatic crinkled phase has $\kappa = 1/\pi$. Thus the crinkled phase in the strong coupling regime has a stiffer bending rigidity than those in the weak coupling regime.

We have presented phase transitions between the internally ordered phase to the disordered liquid phase on the fluctuating membranes using the sine-Gordon renormalization analysis in the absence of the tilt-bond interaction. In this Rapid Communication we have considered two different kinds of internal orders; tilt order and hexatic order. There may be other kinds of internal orders, too. The results developed here can be applied to any two kinds of internal orders. When there are two kinds of internal orders on the fluctuating membrane, there exist two order-disorder transitions and the crumpling transition. These order-disorder transitions occur simultaneously if the coupling constant between two internal orders is in the strong coupling regime, independently if it is in the weak coupling regime. The crumpling transition occurrs when the last existing internal order disappears and the membrane goes into the liquid phase. Thus if the membrane has at least one kind of internal order, the membrane is crinkled, not crumpled, with nonvanishing renormalized bending rigidity.

Inclusion of the tilt-bond interaction in the tilted hexatic Hamiltonian may change the qualitative properties of the fluctuating membranes such as the stiffness of the membranes in the absence of the internal orders or the relations between the local bond orientation and the local tilt direction in the tilted hexatic phase. The tilt-bond interaction can be intepreted as the third internal order with proper modification to make the renormalization analysis complicated but possible. The sine-Gordon approach to the Hamiltonian with the tilt-bond interaction is under investigation.

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